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Generalized Hamiltonian structures for systems in three dimensions with a rescalable constant of motion

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Abstract. The generalized Hamiltonian structures of several three-dimensional dynamical systems of interest in physical applications are considered. In general, Hamiltonians exist only for systems that possess at least one time-independent constant of motion. Systems with only time-dependent constants of motion may sometimes be rescaled and their constant of motion made time-independent. When this is possible, the transformed system may be cast in a generalized Hamiltonian formalism with non-canonical structure functions.

1. Introduction

Recently, Nutku [1] presented a Hamiltonian formulation (in a generalized sense [2]) for an integrable class of the 3D Lotka–Volterra system (LVS). Also recently, Cairó and Feix [3] showed that such structures can always be determined in a closed form for second-order systems with a time-independent constant of motion (CM). Other systems with time-dependent CMs have been expressed in Hamiltonian form via a time-dependent rescaling. In particular, three out of the six known subcases of the Lorenz system that admit a time-dependent CM were recast in Hamiltonian form [4] after rescaling. A subsequent study [5] showed that Hamiltonian structures for 3D vector fields can frequently be constructed if one CM is known. Moreover, when two time-independent CMs are known, the whole problem is reduced to a quadrature.

Recently, three-dimensional systems have become an important topic in mathematical physics, as can be verified in the literature. Some, such as the Euler equations for the rigid-body motion [6] or the ray optic equations in an axisymmetric medium [7], are completely understood and have a well established Hamiltonian form. Others, with a shorter history, deserved considerable attention in recent years (see, for example, [9–12]) mainly because of their importance in applications but also because of their mathematical properties and sometimes unexpected behaviour. One interesting example is the Lorenz system [8] which models the convection of Bérnard cells in a hot fluid layer. Equally important are the 3D LVS that model the interplay of population growth [10] and the behaviour of some physical systems or chemical reactions, the reduced three-wave (RTW) system [11] that describes the interaction of three waves in a conducting medium and the Rabinovich system [12] that also describes three-wave interaction. In this paper, we consider these systems under the conditions (parameter values) where a CM is known and seek their generalized Hamiltonian form. This representation is searched using a formalism [5] developed recently for 3D systems by which the CM becomes the generalized Hamiltonian with (usually) non-canonical structure functions. Since, for the above mentioned systems, the known CMs have

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an explicit dependence on time, a direct application of the procedure proposed in [5] is not possible. This, in fact, should be expected since most of them, like the Lorenz system, support turbulent regimes and, therefore, possess no canonical Hamiltonian globally. To circumvent this problem and obtain an alternative Hamiltonian representation of the same problem, we resort to time rescaling. For several important cases, the rescaling removes the time dependence of the CM and the theory can be applied indirectly. This originates an equivalent formulation of the initial problem which is non-autonomous but admits a generalized Hamiltonian or Poisson structure.

The organization of the paper is as follows. In section 2, we review the basic theory for constructing Poisson structure in 3D. In section 3, we apply the theory to the three rescalable and integrable cases of the Lorenz system. In sections 4–6, we similarly analyse the five integrable cases of the RTW system, the seven instances of the Rabinovich system that admit a CM, and the rescalable time-dependent first integral of the 3D LVS. Since the derivations are quite similar, we present detailed calculations only for some typical cases. In section 7, we present one sample application of the results by studying the nonlinear stability of one example of the Lorenz system that is completely integrable. Finally, in section 8, we present our conclusions.

2. Basic theory

In this section, we review the basic theory of Hamiltonian- or Poisson-structure construction for 3D systems and introduce the concept of associated rescaled systems.

It has been shown recently [5] that all 3D systems

$$\dot{x}^k = v^k(\boldsymbol{x}, t) \qquad k = 1, 2, 3 \tag{1}$$

that admit a time-independent CM† can, in general, be recast in a Hamiltonian form

$$\dot{x}^i = J^{ik}\partial_k H \equiv [x^i, H] \tag{2}$$

where an overdot indicates the derivative with respect to time, J^{ik} are the components of a structure matrix (SM), ∂_k is the partial derivative with respect to x^k and the symbol [,] represents the Poisson bracket. In equations (1) and (2), and throughout the text, the indices run from 1 to 3 and the Einstein summation convention is used unless otherwise specified.

The Hamiltonian formalism requires two ingredients: the Hamiltonian function H and the Poisson bracket which is defined, in a generalized form [2], in terms of the structure functions J^{ik}. The latter must be antisymmetric and satisfy the Jacobi identity which in 3D reads

$$J^{k1}\partial_k J^{23} + J^{k2}\partial_k J^{31} + J^{k3}\partial_k J^{12} = 0. \tag{3}$$

Remark 1. For our purposes, a function $H(\boldsymbol{x})$ of the dynamical variables (x^1, x^2, x^3) is said to be a time-independent CM for (1) iff

$$v^k\partial_k H(\boldsymbol{x}) = 0. \tag{4}$$

If an explicitly time-dependent function $H(\boldsymbol{x}, t)$ satisfies (4) then it is no longer a CM, but may still serve as a Hamiltonian for the system.

† In fact, the theory applies for any function $H(\boldsymbol{x}, t)$ that satisfies equation (4).

The procedure for constructing Hamiltonian structures for 3D systems starts with an otherwise arbitrary anti-symmetric matrix of elements J^{ij} on which we impose the relations

$$v^s = J^{sj}\partial_j H(x) \tag{5}$$

for $s = 1, 2$. This determines two out of the three independent functions in terms of only one of them, say, $J \equiv J^{12}$. The third equality

$$v^3 = J^{3j}\partial_j H(x) \tag{6}$$

is automatically satisfied when $H(x)$ is a first integral for (1).

In particular, we can always choose a labelling for the variables (see remark 2, below) such that

$$\begin{aligned} J^{12} &= J \\ J^{13} &= (v^1 - J\partial_2 H)/(\partial_3 H) \\ J^{23} &= (v^2 + J\partial_1 H)/(\partial_3 H). \end{aligned} \tag{7}$$

The unknown function J is determined from the solution of the first-order linear partial-differential equation (PDE)

$$v^k\partial_k J = AJ + B \tag{8}$$

where A and B are given in terms of the field v^k and the CM by

$$A = \partial_k v^k - (\partial_3 v^k)(\partial_k H)/(\partial_3 H) \tag{9}$$

$$B = (v^1\partial_3 v^2 - v^2\partial_3 v^1)/(\partial_3 H). \tag{10}$$

Equation (8) is the Jacobi identity (3) expressed in terms of J only.

Remark 2. The way A and B depend on $\partial_3 H$ is a consequence of the choice $J = J^{12}$. If in some application $\partial_3 H \equiv 0$, then equations (7), (9) and (10) should undergo a cyclic permutation of the indices $(1 \to 2 \to 3 \to 1)$ in order to trade $\partial_3 H$ with either $\partial_1 H$ or $\partial_2 H$, whichever is non-zero.

The characteristic equations

$$\frac{dx^1}{v^1} = \frac{dx^2}{v^2} = \frac{dx^3}{v^3} = \frac{dJ}{AJ + B} \tag{11}$$

associated with the PDE (8) constitute the working element in our subsequent calculations. However, any particular solution to equation (8) yields a non-trivial SM. Hence, we take the practical approach and solve for the easiest particular solution whenever possible. This will, in general, avoid the full integration of the characteristic equations (11).

In order to apply the theory to systems with a time-dependent first integral, we restrict our attention to rescalable CMs of the form

$$H(x^1, x^2, x^3, t) = P(x^1, x^2, x^3)\exp(st) \tag{12}$$

where $P(x^1, x^2, x^3)$ is a polynomial (or quasi-polynomial) in (x^1, x^2, x^3) and s is a scalar. In practice, this implies no serious limitation since all time-dependent CMs for 3D systems we meet are of this form.

Finally, by rescalable CM, we mean functions of the form (12) for which there exist scalars s_1, s_2 and s_3 such that the substitution

$$x^k = y^k e^{s_k t} \tag{13}$$

(no sum over k) transforms $H(x^1, x^2, x^3, t)$ into $P(y^1, y^2, y^3)$ according to

$$P(x^1 e^{s_1 t}, x^2 e^{s_2 t}, x^3 e^{s_3 t}) = P(x^1, x^2, x^3) e^{-st}. \tag{14}$$

For example, any function of the form (12), where P is homogeneous of order n, becomes time-independent after the transformation (13) with $s_k \equiv s/n$. This transforms the original first integral into a time-independent CM for the corresponding rescaled system.

We shall consider only systems with rescalable first integrals. This choice is justified by the non-existence (at least for the systems treated in this paper) of any alternative strategy that works for systems with a *time-dependent* CM. The rescaling, however, does not deprive the original system of any of its primitive features: we can always re-establish the original representation by the inverse transformation.

After this quick review of the underling theory and the definition of rescalable time-dependent CM, we proceed by analysing separately various 3D systems currently found in the literature. For easier future reference, all results will be summarized in tables.

3. The Lorenz system

The Lorenz system [8]

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= -y + rx - xz \\ \dot{z} &= -bz + xy \end{aligned} \tag{15}$$

where σ, r and b are arbitrary (non-negative) parameters, has deserved considerable attention, mainly in view of its ability to model the onset of chaotic behaviour and the existence of strange attractors.

For appropriate subranges of the parameters, six time-dependent CMs have been determined by some suitable technique [9]. Among these, three are rescalable by transformations of the type (13). This corresponds to case numbers 1, 3 and 5 in Kuś's table [9]. For these special cases, it is possible to rewrite the rescaled system using a generalized Hamiltonian structure.

There is additional freedom in the transformation of the variables when the CM does not depend on some of the dynamical variables $\partial_k H \equiv 0$, for some k. In these cases, we may choose s_k in (13) such that the new vector field is divergenceless. This simplifies the form of A and, hence, the subsequent calculation. For example, in the first case of the Lorenz system $H = (x^2 - 2\sigma z)\exp(2\sigma t)$, the transformed first integral becomes

$$H' = (x^1)^2 - 2\sigma x^3 \tag{16}$$

when the new variables are chosen according to $x^1 = x\exp(\sigma t)$, $x^2 = y$ and $x^3 = z\exp(2\sigma t)$. However, it is more advantageous for later calculations to simultaneously transform x^2 by $x^2 = y\exp(t)$. In this case, the rescaled CM remains the same but the new Lorenz vector field (notice that here $b = 2\sigma$) becomes divergenceless

$$\begin{aligned} v^1 &= \sigma x^2 e^{(\sigma-1)t} \\ v^2 &= x^1(r - x^3 e^{-2\sigma t})e^{(1-\sigma)t} \\ v^3 &= x^1 x^2 e^{(\sigma-1)t}. \end{aligned} \tag{17}$$

To illustrate the general procedure, we now present the calculation details of the Poisson structure for the rescaled Lorenz system with CM (16). For the rescaling listed in row LOR(1) of table 1 and using (9) and (10), we obtain $A \equiv 0$ and

$$B = \tfrac{1}{2}x^1 x^2 \exp(-2\sigma t). \tag{18}$$

Hence, in the new variables, one of the characteristic equations (11), in particular,

$$\frac{dx^3}{v^3} = \frac{dJ}{AJ + B} \tag{19}$$

becomes

$$\frac{dJ}{dx^3} = \tfrac{1}{2}e^{(1-3\sigma)t} \tag{20}$$

which is readily integrated for the particular solution

$$J = \tfrac{1}{2}x^3 e^{(1-3\sigma)t}. \tag{21}$$

The whole calculation is completed by substituting J into equations (7) which yields the following elements for the SM:

$$\begin{aligned} J^{12} &= \tfrac{1}{2}x^3 e^{(1-3\sigma)t} \\ J^{13} &= -\tfrac{1}{2}x^2 e^{(\sigma-1)t} \\ J^{23} &= -\frac{r}{2\sigma}x^1 e^{(1-\sigma)t}. \end{aligned} \tag{22}$$

This determines a generalized Hamiltonian or Poisson structure for the rescaled Lorenz system restricted to the parameter values given in case 1 in Kuś's table. This Poisson structure is completely specified by the Hamiltonian $H' = (x^1)^2 - 2\sigma x^3$ and the structure functions in (22). All the above results are regrouped in tables 1–3 under rows LOR(1).

To proceed, we treat the third case in Kuś's table, i.e. the case where $H = (y^2 + z^2)\exp(2t)$. Now, we transform the variables according to the line labelled LOR(3) in table 1 and obtain the time-independent rescaled CM $H' = (x^2)^2 + (x^3)^2$. Under this transformation, the vector field (LOR(3) in table 2) becomes divergenceless and implies A and B given by

$$A = \frac{x^1 x^2}{x^3}e^{-\sigma t} \qquad \text{and} \qquad B = -\tfrac{1}{2}A\sigma e^{(\sigma-1)t}. \tag{23}$$

Table 1. Transformations and Rescaled Constants of Motion.

Case	x^1	x^2	x^3	Hamiltonians
LOR(1)	$x\exp(\sigma t)$	$y\exp(t)$	$z\exp(2\sigma t)$	$(x^1)^2 - 2\sigma x^3$
LOR(3)	$x\exp(\sigma t)$	$y\exp(t)$	$z\exp(t)$	$(x^2)^2 + (x^3)^2$
LOR(5)	$x\exp(t)$	$y\exp(t)$	$z\exp(t)$	$(x^2)^2 - r(x^1)^2 + (x^3)^2$
RTW(1)	x	y	$z\exp(2t)$	$x^3(2x^2 - \delta)$
RTW(2)	$x\exp(t)$	$y\exp(t)$	$z\exp(2t)$	$(x^1)^2 + (x^2)^2 + x^3$
RTW(3)	$x\exp(t)$	$y\exp(t)$	$z\exp(2t)$	x^2x^3
RTW(4)	$x\exp(-\gamma t)$	$y\exp(-\gamma t)$	$z\exp(2t)$	x^2x^3
RTW(5)	$x\exp(2t)$	$y\exp(2t)$	$z\exp(2t)$	$(x^1)^2 + (x^2)^2 + 2x^2x^3/\delta$
RAB(1)	$x\exp(\nu t)$	$y\exp(\nu t)$	$z\exp(2\nu t)$	$(x^1)^2 + (x^2)^2 - 4hx^3$
RAB(2)	$x\exp(\nu t)$	$y\exp(\nu t)$	$z\exp(\nu t)$	$(x^1)^2 - (x^2)^2 - 2(x^3)^2$
RAB(3)	$x\exp(\nu t)$	$y\exp(\nu t)$	$z\exp(\nu_3 t)$	$(x^1)^2 + (x^2)^2$
RAB(4)	$x\exp(\nu_1 t)$	y	z	$(x^2)^2 + (h - x^3)^2$
RAB(5)	x	$y\exp(\nu_2 t)$	z	$(x^1)^2 - (h + x^3)^2$
RAB(6)	$x\exp(\nu_1 t)$	$y\exp(\nu_3 t)$	$z\exp(\nu_3 t)$	$(x^2)^2 + (x^3)^2$
RAB(7)	$x\exp(\nu_3 t)$	$y\exp(\nu_2 t)$	$z\exp(\nu_3 t)$	$(x^1)^2 - (x^3)^2$
LVS(1)	$x\exp(-a_1 t)$	$y\exp(-a_2 t)$	$z\exp(-a_3 t)$	$(x^1)^\alpha (x^2)^\beta (x^3)^\gamma$

Table 2. Rescaled fields.

Case	v^1	v^2	v^3
LOR(1)	$\sigma x^2\exp[(\sigma-1)t]$	$x^1[r - x^3\exp(-2\sigma t)]\exp[(1-\sigma)t]$	$x^1x^2\exp[(\sigma-1)t]$
LOR(3)	$\sigma x^2\exp[(\sigma-1)t]$	$-x^1x^3\exp(-\sigma t)$	$x^1x^2\exp(-\sigma t)$
LOR(5)	x^2	$rx^1 - x^1x^3\exp(-t)$	$x^1x^2\exp(-t)$
RTW(1)	$\delta x^2 + x^3\exp(-2t) - 2(x^2)^2$	$-\delta x^1 + 2x^1x^2$	$-2x^1x^3$
RTW(2)	$\delta x^2 + (x^3 - 2(x^2)^2)\exp(-t)$	$-\delta x^1 + 2x^1x^2\exp(-t)$	$-2x^1x^3\exp(-t)$
RTW(3)	$(x^3 - 2(x^2)^2)\exp(-t)$	$2x^1x^2\exp(-t)$	$-2x^1x^3\exp(-t)$
RTW(4)	$x^3\exp[-(\gamma+2)t] - 2(x^2)^2\exp(\gamma t)$	$2x^1x^2\exp(\gamma t)$	$-2x^1x^3\exp(\gamma t)$
RTW(5)	$\delta x^2 + x^3 - 2(x^2)^2\exp(-2t)$	$-\delta x^1 + 2x^1x^2\exp(-2t)$	$-2x^1x^3\exp(-2t)$
RAB(1)	$x^2[h + x^3\exp(-2\nu t)]$	$x^1[h - x^3\exp(-2\nu t)]$	x^1x^2
RAB(2)	$x^2[h + x^3\exp(-\nu t)]$	$x^1[h - x^3\exp(-\nu t)]$	$x^1x^2\exp(-\nu t)$
RAB(3)	$x^2x^3\exp(-\nu_3 t)$	$-x^1x^3\exp(-\nu_3 t)$	$x^1x^2\exp[(\nu_3 - 2\nu)t]$
RAB(4)	$x^2(h + x^3)\exp(\nu_1 t)$	$x^1(h - x^3)\exp(-\nu_1 t)$	$x^1x^2\exp(-\nu_1 t)$
RAB(5)	$x^2(h + x^3)\exp(-\nu_2 t)$	$x^1(h - x^3)\exp(\nu_2 t)$	$x^1x^2\exp(-\nu_2 t)$
RAB(6)	$x^2x^3\exp[(\nu_1 - 2\nu_3)t]$	$-x^1x^3\exp(-\nu_1 t)$	$x^1x^2\exp(-\nu_1 t)$
RAB(7)	$x^2x^3\exp(-\nu_2 t)$	$-x^1x^3\exp[(\nu_2 - 2\nu_3)t]$	$x^1x^2\exp(-\nu_2 t)$
LVS(1)	$x^1\sum_k b_{1k}\exp(a_k t)x^k$	$x^2\sum_k b_{2k}\exp(a_k t)x^k$	$x^3\sum_k b_{3k}\exp(a_k t)x^k$

The ratio of B and A is independent of the dynamical variables and consequently

$$J \equiv -\frac{B}{A} = \frac{1}{2}\sigma e^{(\sigma-1)t} \tag{24}$$

satisfies equation (11) as can be verified directly. This provides a particular solution to (8) and completes the derivation of the Poisson structure for case 3 in Kuś's table. The detailed form of the transformation, the rescaled vector field, the Hamiltonian and the elements of the SM are summarized in row LOR(3) in tables 1–3.

The last rescalable CM in the Lorenz system is $(y^2 - rx^2 + z^2)\exp(-2t)$ (case 5 in Kuś's table) and can be treated similarly. The coordinate transformation, the rescaled CM and, hence, the Hamiltonian, the new vector field as well as the final form of the structure functions are again listed in the same tables under row LOR(5).

Table 3. The structure functions.

System	J^{12}	J^{13}	J^{23}
LOR(1)	$\frac{1}{2}x^3\exp[(1-3\sigma)t]$	$\frac{1}{2}x^2\exp[(\sigma-1)t]$	$\frac{1}{2}(r/\sigma)x^1\exp[(1-\sigma)t]$
LOR(3)	$\frac{1}{2}\sigma\exp[(\sigma-1)t]$	0	$-\frac{1}{2}x^1\exp(-\sigma t)$
LOR(5)	$\frac{1}{2}$	0	$-\frac{1}{2}x^1\exp(-t)$
RTW(1)	$\frac{1}{2}\exp(-2t)$	$-x^2$	x^1
RTW(2)	$-x^2\exp(-t)$	$\delta x^2+x^3\exp(-t)$	$-\delta x^1$
RTW(3)	$\exp(-t)$	$-2x^2\exp(-t)$	$2x^1\exp(-t)$
RTW(4)	$\exp[-(2+\gamma)t]$	$-2x^2\exp(\gamma t)$	$2x^1\exp(\gamma t)$
RTW(5)	$\delta/2$	$-\delta x^2\exp(-2t)$	$\delta x^1\exp(-2t)$
RAB(1)	$\frac{1}{2}x^3\exp(-2\nu t)$	$-\frac{1}{4}x^2$	$-\frac{1}{4}x^1$
RAB(2)	$-\frac{1}{2}h$	$-\frac{1}{4}x^2\exp(-\nu t)$	$\frac{1}{4}x^1\exp(-\nu t)$
RAB(3)	$\frac{1}{2}x^3\exp(-\nu_3 t)$	$-\frac{1}{2}x^2\exp[(\nu_3-2\nu)t]$	0
RAB(4)	$h\exp(\nu_1 t)$	$\frac{1}{2}x^2\exp(\nu_1 t)$	$-\frac{1}{2}x^1\exp(-\nu t)$
RAB(5)	$-h\exp(\nu_2 t)$	$-\frac{1}{2}x^2\exp(-\nu_2 t)$	$\frac{1}{2}x^1\exp(\nu_2 t)$
RAB(6)	0	$\frac{1}{2}x^2\exp[(\nu_1-2\nu_2)t]$	$-\frac{1}{2}x^1\exp(-\nu_1 t)$
RAB(7)	0	$-\frac{1}{2}x^2\exp(-\nu_2 t)$	$\frac{1}{2}x^1\exp[(\nu_2-2\nu_3)t]$
LVS(1)	Q	$1/(\gamma x^2)(Uu^1/H-x^3Q)$	$1/(\gamma x^1)(Uu^2/H+\alpha x^3 Q)$

4. The RTW interaction

The (RTW) consist of the equations [11]

$$\begin{aligned}\dot{x}&=\gamma x+\delta y+z-2y^2\\ \dot{y}&=\gamma y-\delta x+2xy\\ \dot{z}&=-2z(x+1)\end{aligned}\qquad(25)$$

where x. y and z denote the amplitudes of interacting waves in appropriate units and the parameters δ and γ measure detuning from synchronism. The system is known to admit five CMs [12] which we list in table 4 for future reference. It is readily seen in table 4 that here, unlike in the Lorenz system, all CMs are rescalable and, as a consequence, Hamiltonian structures can easily be determined for their rescaled versions. The adequate transformations, the rescaled fields, the CM and, therefore, the Hamiltonians and associated structure functions are listed in table 1–3, rows RTW(i; $i=1\ldots5$).

Table 4. Reduced three-wave interaction.

Case	γ	δ	CM
RTW(1)	$\gamma=0$	$\forall$	$z(y-\delta/2)\exp(2t)$
RTW(2)	$\gamma=-1$	$\forall$	$[(x)^2+(y)^2+z]\exp(2t)$
RTW(3)	$\gamma=-1$	$\delta=0$	$zy\exp(3t)$
RTW(4)	$\forall$	$\delta=0$	$zy\exp[(2-\gamma)t]$
RTW(5)	$\gamma=-2$	$\forall$	$[(x)^2+(y)^2+2yz/\delta]\exp(4t)$

The derivation of the basic component $J=J^{12}$ of the SM parallels the derivation performed in the analysis of the Lorenz system in section 3. In fact, for cases RTW(1,3–5), the function J is obtained in a way similar to that in LOR(3), basically, because in these cases all the ratios B/A are independent of the dynamical variables. The case RTW(2) is tackled in a manner similar to LOR(1).

5. The Rabinovich system

The Rabinovich system is also a three-wave interaction system [11] and is described by the equations

$$\begin{aligned} \dot{x} &= hy - \nu_1 x + yz \\ \dot{y} &= hx - \nu_2 y - xz \\ \dot{z} &= -\nu_3 z + xy \end{aligned} \tag{26}$$

where x, y and z are wave amplitudes, ν_1, ν_2 and ν_3 are damping rates and h is a constant proportional to the driving amplitude of the feeder wave. This system has also been scrutinized [12] and a total of seven CMs have been identified. These results are listed in table 5 in which the labels RAB identify the different cases and corresponding parameter ranges.

Table 5. Rabinovich system.

Case	h	ν_1	ν_2	ν_3	CM
RAB(1)	$\forall$	$\nu > 0$	$\nu > 0$	$2\nu > 0$	$[(x)^2 + (y)^2 - 4hz]\exp(2\nu t)$
RAB(2)	$\forall$	$\nu > 0$	$\nu > 0$	$\nu > 0$	$[(x)^2 - (y)^2 - 2(z)^2]\exp(2\nu t)$
RAB(3)	0	$\nu > 0$	$\nu > 0$	$\forall$	$[(x)^2 + (y)^2]\exp(2\nu t)$
RAB(4)	$\forall$	$\forall$	0	0	$(y)^2 + (h - z)^2$
RAB(5)	$\forall$	0	$\forall$	0	$(x)^2 - (h + z)^2$
RAB(6)	0	$\forall$	$\nu_2 = \nu_3$	$\forall$	$[(y)^2 + (z)^2]\exp(2\nu_3 t)$
RAB(7)	0	$\nu_1 = \nu_3$	$\forall$	$\forall$	$[(x)^2 - (z)^2]\exp(2\nu_3 t)$

The CMs in RAB(4,5) are time-independent and one would, therefore, expect them to solve the associated characteristic equations directly without first passing by the rescaling stage. Unfortunately, however, the resulting equations are difficult to handle and we were unable to tackle them without first rescaling the variables according to rows RAB(4,5) of table 2. The calculations become easier in rescaled coordinates essentially because the new vector fields are divergenceless. The complete derivation of the basic component J of the SM, for cases RAB(1,2,4,5) in rescaled coordinates, was accomplished following a sequence of steps that parallel the treatment of LOR(3). The results are registered in tables 1–3 as usual.

To complete this section, we present some details of the calculations for RAB(3). For this case, $\partial_3 H \equiv 0$ and (7), (9) and (10) need to undergo one cyclic permutation of the indexes. This implies $J = J^{23}$ and

$$A = \partial_k v^k - \frac{(\partial_1 v^k)(\partial_k H)}{\partial_1 H} \tag{27}$$

$$B = \frac{v^2 \partial_1 v^3 - v^3 \partial_1 v^2}{\partial_1 H}. \tag{28}$$

The CM in RAB(3) is not dependent on z. However, when this variable is conveniently rescaled using $s_3 \equiv \nu_3$, we obtain $B \equiv 0$ and (8) becomes homogeneous. In such cases, $J = 0$ is trivially the simplest solution. This does not imply a trivial solution for the SM, as can be checked in table 3 where the remaining J^{ik} are listed in row RAB(3).

Finally RAB(7), the last case in table 5, also has $B \equiv 0$ in rescaled coordinates and, hence, $J = 0$. Again, the complete results of the calculations are collected on rows RAB(7) in tables 1–3.

6. The LVS

Finally, we come to the LVS where the solution to equation (11) is the most difficult to calculate. The LVS and special subsystems are quite useful in modelling many physical, chemical and biological processes and are defined, in their most general form, by

$$\begin{aligned}\dot{x} &= x(a_1 + b_{11}x + b_{12}y + b_{13}z)\\ \dot{y} &= y(a_2 + b_{21}x + b_{22}y + b_{23}z)\\ \dot{z} &= z(a_3 + b_{31}x + b_{32}y + b_{33}z).\end{aligned} \tag{29}$$

We note in passing that both the *abc* [13] and the May–Leonard systems [14] are special cases of the LVS.

Cairó and Feix [3] have several CMs for N-dimensional LVS. In 3D, one of these is rescalable. This, in particular, reads

$$H = x^{\alpha} y^{\beta} z^{\gamma} \mathrm{e}^{-st} \tag{30}$$

where α, β, γ and s are given by

$$\alpha = b_{22}b_{31} - b_{21}b_{32} \qquad \beta = b_{11}b_{32} - b_{12}b_{31}$$

$$\gamma = b_{12}b_{21} - b_{11}b_{22} \qquad s = a_1\alpha + a_2\beta + a_3\gamma$$

and $\det(b_{ij}) = 0$. Under the additional constraint $s = 0$, the H becomes time-independent and the system possesses a generalized Hamiltonian structure with five free parameters [5]. For arbitrary s, we use rescaling according to what is listed in row LVS(1) of table 1. The rescaled vector field and time-independent CM (hereafter represented by H') are presented in rows LVS(1) of tables 1–3.

We now present some details of the calculation. First, introduce the compact notation

$$v^k \equiv x^k u^k \qquad \text{where} \qquad u^k \equiv \sum_j b_{kj} \mathrm{e}^{a_j t} x^j \tag{31}$$

and $(x^1, x^2, x^3) = (x, y, z)$, summation over k is not implied and express A and B, necessary in equation (8), as

$$A = u^1 + u^2 + \sum_j b_{jj} \mathrm{e}^{a_j t} x^j \tag{32}$$

$$B = U \exp(a_3 t)(b_{23}u^1 - b_{13}u^2)/(\gamma H') \tag{33}$$

where $U \equiv x^1x^2x^3$. Unfortunately, there is no simple solution for J when A and B are given by (32) and (33). The most general situation that we can deal with is obtained by imposing one additional condition on the coefficient b_{ij}, namely, equality (41) below. This, however, still leaves *seven* free parameters and, therefore, includes more general systems than those treated by Haas and Goedert with *five* or the *abc* system solved by Nutku with *four* free parameters.

Elementary algebraic rules applied on the characteristic equation (11) imply

$$\frac{\mathrm{d}U}{U(u^1 + u^2 + u^3)} = \frac{\mathrm{d}J}{AJ + B}. \tag{34}$$

To further simplify the notation, we now introduce the symbol

$$\theta_{ij} = b_{ij}\exp(a_j t) \tag{35}$$

with no sum over j, and make the substitution

$$(u^1 + u^2 + u^3) = A + (\theta_{31} - \theta_{11})x^1 + (\theta_{32} - \theta_{22})x^2.$$

This transforms (34) into

$$\frac{\mathrm{d}U}{U[A + (\theta_{31} - \theta_{11})x^1 + (\theta_{32} - \theta_{22})x^2]} = \frac{\mathrm{d}J}{AJ + B}. \tag{36}$$

Equation (36) allows us to recast the characteristic equation into the form

$$\frac{\mathrm{d}x^k}{v^k} = \frac{\mathrm{d}[J - \lambda U/(\gamma H')]}{A[J - \lambda U/(\gamma H')] - (B'U)/(\gamma H')} \tag{37}$$

where λ is an arbitrary function of time to be chosen later at convenience and

$$B' = [\lambda(\theta_{31} - \theta_{11}) - \theta_{23}\theta_{11} + \theta_{13}\theta_{21}]x^1 + [\lambda(\theta_{32} - \theta_{22}) - \theta_{23}\theta_{12} + \theta_{13}\theta_{22}]x^2. \tag{38}$$

Equation (37) is difficult to solve in its general form. We tackle the problem by imposing the extra condition $B' \equiv 0$. As can be easily seen, under this additional constraint,

$$J = \frac{\lambda U}{\gamma H'} = \frac{\lambda}{\gamma}(x^1)^{1-\alpha}(x^2)^{1-\beta}(x^3)^{1-\gamma} \tag{39}$$

is a solution to (37) and, therefore, to the fundamental condition (8).

The condition on B' determines the value of the arbitrary function

$$\lambda = \frac{\mathrm{e}^{a_3 t}(b_{23}b_{11} - b_{13}b_{21})}{b_{31} - b_{11}} \tag{40}$$

and implies, in addition, that

$$(b_{32} - b_{22})(b_{23}b_{11} - b_{13}b_{21}) = (b_{31} - b_{11})(b_{23}b_{12} - b_{13}b_{22}). \tag{41}$$

This yields

$$J = Q \equiv \frac{b_{23}b_{11} - b_{13}b_{21}}{\gamma(b_{31} - b_{11})}\mathrm{e}^{a_3 t}(x^1)^{1-\alpha}(x^2)^{1-\beta}(x^3)^{1-\gamma} \tag{42}$$

which determines, through equations (7), the form of the SM. Their final forms are listed in row LVS(1) of table 3.

Note that in order to solve for J, we needed to impose condition (41) on the coefficient b_{ij} in addition to $\det(b_{ij}) = 0$. Hence, there remains ten out of the original coefficients in the LVS. Moreover, three coefficients can always be set to one by rescaling, leaving, therefore, seven arbitrary parameters.

7. Nonlinear stability analysis

The procedure for constructing the generalized Hamiltonian structure of the 3D systems can be applied twice whenever the system admits two functionally-independent CM that are either time-independent or can be rescaled into time-independent CM. When this is possible, we may eventually come up with bi-Hamiltonian structures. These become more interesting later mainly because they imply complete integrability and frequently provide additional means to study the system in more detail. An interesting problem to examine when a system possesses a bi-Hamiltonian structure is the nonlinear stability of its critical points. Although this is not in the original scope of this study, we present here a quick analysis of the nonlinear stability of a special case of the Lorenz system that admits a bi-Hamiltonian structure. Other systems in this paper admit, similarly, a second CM and could also be inspected for stability.

To illustrate the possibility of nonlinear stability analysis, we consider the Lorenz system with $\sigma = \frac{1}{2}$, $b = 1$ and $r = 0$. For these parameter values, the system possesses two CMs, H_1 and H_2

$$H_1 = (y^2 + z^2)\exp(2t) \qquad H_2 = (x^2 - z)\exp(t). \tag{43}$$

If we rescale the dynamical variables according to

$$\bar{x} = x\exp(t/2) \qquad \bar{y} = y\exp(t) \qquad \bar{z} = z\exp(t) \tag{44}$$

and transform the time by using

$$\bar{t} = -2\exp(-t/2) \tag{45}$$

we obtain the equivalent system

$$\bar{x}' = \tfrac{1}{2}\bar{y}\bar{y}' = -\bar{x}\bar{z}\bar{z}' = \bar{x}\bar{y} \tag{46}$$

where the $'$ means derivative with respect to the new time. The rescaled system (46) has two time-independent CMs, $\bar{H}_1 = (\bar{y}^2 + \bar{z}^2)$ and $\bar{H}_2 = (\bar{x}^2 - \bar{z})$, and fixed points on $r_1 = (0, 0, z_e)$ and $r_2 = (x_e, 0, 0)$. Notice that, except for the origin, the fixed points of the rescaled system are not fixed points in the original system.

We now proceed with the stability analysis according to the procedure proposed in [7]. For the CM $\bar{H}_1$, we find the structure functions

$$\begin{aligned} J^{12} &= -J^{21} = \tfrac{1}{4} \\ J^{13} &= -J^{31} = 0 \\ J^{23} &= -J^{32} = -\tfrac{1}{2}\bar{x} \end{aligned} \tag{47}$$

and the associated Casimir $C_1 \equiv \bar{H}_2$. The energy Casimir function can, therefore, be written

$$H(C) = \bar{H}_1 + F(\bar{H}_2) = \bar{y}^2 + \bar{z}^2 + F(\bar{x}^2 - \bar{z}) \tag{48}$$

where F is any arbitrary function of its argument. The first variation of $H(C)$ is given by

$$DH(C)\delta r = 2\bar{y}\,\mathrm{d}\bar{y} + 2\bar{z}\,\mathrm{d}\bar{z} + (2\bar{x}\,\mathrm{d}\bar{x} - \mathrm{d}\bar{z})F' \tag{49}$$

where $F' = \mathrm{d}F/\mathrm{d}C$. For r_1, that is for z_e on the $\bar{z}$-axis, $DH(C)\delta r$ is zero iff $F'(-z_e) = 2z_e$.

To study the formal stability (in finite dimensions formal stability implies nonlinear stability), we calculate the second variation of $H(C)$

$$D^2H(C)(\delta r)^2 = 2(\mathrm{d}\bar{y})^2 + 2(\mathrm{d}\bar{z})^2 + 2(\mathrm{d}\bar{x})^2F' + (2\bar{x}\,\mathrm{d}\bar{x} - \mathrm{d}\bar{z})^2F''. \tag{50}$$

At equilibrium, we obtain

$$D^2H(C)(\delta r)^2 = 2(\mathrm{d}\bar{y})^2 + 4z_e(\mathrm{d}\bar{x})^2 + (2 + F''(-z_e))(\mathrm{d}\bar{z})^2 \tag{51}$$

which is positive definite iff $z_e > 0$ and $2 + F''(-z_e) > 0$. For example, the function $F(\bar{z}) = 2z_e\bar{z} + (\bar{z} + z_e)^2$, among others, satisfies the conditions on F' and F'' over all positive z_e. Therefore, the positive $\bar{z}$-axis is nonlinearly stable under the action of the rescaled field (46).

The equilibrium r_2 can be analysed in a similar way. The annihilation of the first variation and the positive definiteness of the second variation imply $F'(x_e^2) = 0$ and $F''(x_e^2) > 0$. In this case, $F(\bar{z}) = (\bar{z} - x_e^2)^2$ satisfies the requirements and consequently r_2 is nonlinearly stable for arbitrary x_e.

Other systems analysed in this paper can be studied in the same way. In particular, the RTW system for $\delta = 0$ and $\gamma = -1$ has a rescaled version with equilibria $(0, y_e, 2y_e^2)$ and $(x_e, 0, 0)$. When $y_e \neq 0$, the first equilibrium is nonlinearly stable in the formulation of the corresponding Hamiltonian. The second equilibrium cannot be classified by use of the same Hamiltonian and would require a second CM to resolve its stability property.

8. Conclusions

In this paper, we have shown that several 3D systems of interest in physics or biology, notorious for their peculiar behaviour (e.g. existence of an associated chaotic regime or strange attractors), can be endowed with a generalized Hamiltonian or Poisson structure if we rescale the dynamical variable appropriately. This possibility seems to open a new avenue in the study of such dynamical systems because the rescaling transformation can always be reversed, a procedure that 'projects' the solutions to the new equations back to the original phase space. In our analysis, we considered various examples, currently found in the literature, for their importance in relation to virtual applications. Our choice, however, was mainly dictated by the fact that, for some parameter regimes, these systems are known to possess time-dependent CMs. This *per se* implies some degree of intrinsic symmetry which is reminiscent of a Hamiltonian structure. Possible applications related to these Poisson structures, like the stability analysis of the rescaled version of these systems, are open questions that were only touched upon in section 7 to show their possibility. A complete study of this and other related issues is not in the scope of this paper.

Finally, we remark that some of the systems treated in rescaled form here have already previously been treated in the same spirit (finding their associated generalized Hamiltonian or Poisson structure). This was the case mainly for the integrable LVS, which was first shown to admit a Hamiltonian structure by Nutku and whose results were recently generalized by Haas and Goedert. The rescaled Lorenz system had also been considered before. Here, we generalized these results. In particular, we presented the explicit calculation for the structure functions J^{ij} of all the rescalable CMs of the Lorenz (three), the rescalable CM of the 3D LVS and all the known CMs of the RTW (five) and Rabinovich (seven) systems.

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