MHD Equations for Quantum Plasmas

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Abstract. We discuss the quantum magnetohydrodynamic (QMHD) model for quantum plasmas. In the infinite conductivity limit, QMHD differ from classical MHD in view of the presence of an extra dispersive term in the force equation. The quantum term is proportional to a non-dimensional parameter H. We derive conditions for unavoidable quantum effects in plasmas and for large H values. These results may be valuable for the identification of concrete quantum plasma systems where the QMHD equations are useful.

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INTRODUCTION

Usually, classical physics is sufficient to describe the behavior of plasmas. However, in some plasmas under extreme conditions like in dense astrophysical plasmas (white dwarfs), in some laser plasmas, in the electron gas in metals or in ultra-small electronic devices, quantum mechanics have to be taken into account. This has motivated a renewed interest in modeling quantum effects in plasmas (see reference 1 for a general review). Some new quantum beam instabilities in plasmas have been detected [2]-[5], which may however be attenuated if statistical effects are incorporated [6]. Other developments include the construction of a quantum Zakharov system [7] describing the nonlinear interaction of quantum Langmuir and ion-acoustic waves. In this case, quantum effects points to the attenuation of modulational instabilities, in comparison with classical plasmas. However, for a coherent background electric field and not just a monochromatic wave, modulational instabilities have been shown to be enhanced by quantum interactions [8]. Further recent developments deal with the propagation of ion-acoustic waves in quantum plasmas [9].

Recently [10] a modification of magnetohydrodynamics has been proposed, incorporating a quantum term, which is reminiscent of the Bohm potential. This quantum magnetohydrodynamic (QMHD) model was derived according to the following steps: a) starting from the Wigner equations for a plasma under a magnetic field, define fluid variables like mass density and velocity just like in the classical fluid model for plasmas. In this context, the Wigner equations plays the rôle of the Vlasov equation in classical plasmas; b) consider a two-species plasma. In the transport equations, introduce phenomenological terms describing collisions; c) proceed just like in classical MHD, combining the two species fluid defining global fluid variables as mass and charge densities and velocity field; d) introduce some equation of state in order to obtain a closed set of equations. The detailed derivation of the QMHD equations is shown in reference [10]. The purpose of the present communication is simply a more complete discussion on the conditions of validity of QMHD, as well as the identification of physical contexts where quantum effects in magnetohydrodynamics can be noticeable. Our approach relies on the construction of some dimensionless parameters related to the magnitude of quantum effects in MHD plasmas.

QUANTUM PARAMETERS

Consider a two-species quantum plasma with ion mass and charge m_i and e as well as electron mass and charge m_e and $-e$ respectively. In the infinite conductivity limit, the (ideal) QMHD equations read [10]

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{u}) = 0\,, \tag{1}$$

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\frac{v_s^2}{v_a^2}\nabla\rho + (\nabla\times\mathbf{B})\times\mathbf{B} + \frac{H^2\rho}{2}\nabla\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right)\,, \tag{2}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B})\,, \tag{3}$$

where

$$H = \frac{\hbar\Omega_i}{\sqrt{m_e m_i}\,v_a^2} \tag{4}$$

is a non-dimensional parameter measuring the relevance of quantum effects. In the system (1-3), ρ_m, $\mathbf{B}$ and $\mathbf{u}$

are the mass density, the magnetic field and the velocity field, normalized to the equilibrium mass density ρ_0, the equilibrium magnetic field intensity B_0 and the Alfvén velocity $v_a = (B_0^2/(\mu_0\rho_0))^{1/2}$, respectively. In addition, v_s is the adiabatic speed of sound in the fluid and the parameter H is defined also in terms of the ion cyclotron frequency $\Omega_i = eB_0/m_i$. Finally, time is normalized by Ω_i^{-1} and the spatial coordinates by v_a/Ω_i.

In comparison to classical ideal MHD, the ideal QHMD equations differ only in the presence of the last term in Equation (2). This contribution is reminiscent of the Bohm potential in the hydrodynamic formulation of quantum mechanics and may be valuable for the description of quantum phenomena like tunneling. For $H \to 0$, classical MHD is recovered. On the other hand, for H of order unity, we expect that quantum effects plays a significant rôle. However, the quantum term may be negligible even for large H, for a sufficiently slowly varying density, in view of the presence of third-order spatial derivatives. For simplicity, we disregard this possibility here. Hence, we suppose that $H > 1$ is a sufficient sign for the existence of relevant quantum corrections to MHD.

Reference 10 includes the derivation of exact magnetostatic solutions for the ideal QMHD equations. Here we limit ourselves to the discussion of some conditions for the relevance of quantum effects. As discussed elsewhere [1], one instance where quantum mechanics is unavoidable is when the temperature T is lower than the Fermi temperature T_F of the system. In this case, the fermionic character of the charge carriers manifests itself via the Pauli exclusion principle. Alternatively, the condition $T < T_F$ can be reformulated assuming a large number of particles in a box of length λ_q, where λ_q is the de Broglie wavelength. Both conditions imply a significant overlap of the wave functions of the charge carriers. The condition $T < T_F$ can be more easily fulfilled by the electrons due to their lower mass in comparison to ions, giving

$$T < T_F = \frac{\hbar^2}{2m_e\kappa_B}(3\pi^2)^{2/3} n_0^{2/3} \,, \tag{5}$$

where n_0 is the equilibrium particle density and the other symbols have their usual meaning. Now considering the condition $n_0\lambda_q^3 > 1$ when there is a large number of particles in a box of wavelength $\lambda_d = \hbar/(m_e v_T)$, where $v_T = (\kappa_B T/m_e)^{1/2}$ is the electron thermal velocity, we get

$$T < \frac{\hbar^2 n_0^{2/3}}{m_e\kappa_B} \,, \tag{6}$$

a condition very similar to (5), except for a numerical factor. In all situations, quantum mechanics is necessary for large densities and/or low temperatures.

On the other hand, a necessary condition for the validity of fluid models is the occurrence of a large rate of collisions, so that the plasma may be considered as a continuous medium. Assuming that the plasma is degenerate, as discussed in the reference 1, this implies large values of a quantum graininess parameter g_q defined as the ratio of potential and kinetic energies. This condition for large rate of collisions may be expressed as

$$g_q = \left(\frac{\hbar\omega_p}{E_F}\right)^2 > 1 \,, \tag{7}$$

where $\omega_p = (n_0 e^2/(m_e\varepsilon_0))^{1/2}$ is the electron plasma frequency and $E_F = (3\pi^2)^{2/3}\hbar^2 n_0^{2/3}/(2m_e)$ is the electron Fermi energy. For the relevance of the QMHD model, the inequality (7) have to be satisfied, for otherwise there will be the need of a kinetic (Wigner) description. Notice that the inequality $g_q > 1$ is more easily fulfilled for not too high-density quantum plasmas, since for larger densities Pauli blocking tends to avoid collisions.

In conclusion, we have derived three conditions for quantum corrections for MHD. One condition, $H > 1$, yields a large quantum correction term in the force equation of QMHD equations. Other condition, $T/T_F < 1$, points to the relevance of quantum statistics (degenerate plasma). Finally, $g_q > 1$ is an heuristic necessary condition for the validity of a fluid description. Considering a hydrogen plasma, using MKS units and defining

$$x = \log_{10} n_0 \,, \quad y = \log_{10} T \,, \quad z = \log_{10} B_0 \,, \tag{8}$$

we express the conditions for the relevance of the QMHD model according to

$$x < 32.1 \quad \text{(fluid description)} \tag{9}$$

$$y < -14.4 + 2x/3 \quad \text{(degenerate plasma)} \,, \tag{10}$$

$$z < -30.3 + x \quad (\text{large } H) \,. \tag{11}$$

Inequalities (9-11), also imply $y < 7.0$ and $z < 1.8$. Therefore, the temperature must not exceed $T = 10^7 K$ and the ambient magnetic field must not exceed $B_0 = 63.1T$. The constraint on the temperature also assures that the plasma is non relativistic, since $\kappa_B T/(m_e c^2) < 1.7 \times 10^{-3}$.

Equation (9) (the condition for a degenerate plasma) is a sufficient condition for the manifestation of quantum mechanics, Equations (10) and (11) are necessary conditions, for the validity of a fluid description and for large values of H in the QMHD equations, respectively. Other conditions for the relevance of quantum effects may be also devised [11]. For instance, quantum mechanics would be necessary if the de Broglie wavelength λ_q exceeds the electron Debye length $\lambda_d = (\varepsilon_0\kappa_B T/(n_0 e^2))^{1/2}$. However, in our notation,

$$\lambda_q > \lambda_d \Rightarrow y < x/2 - 9.4 < 6.7 \,, \tag{12}$$

the last inequality following from the constraint $x < 32.1$. For large densities ($x > 30.0$), the inequality (12) is less stringent than the second inequality in (10). However, there is not a big numerical difference between the two criteria.

A later possibility for the appearance of quantum effects in plasmas is for

$$\frac{\hbar\Omega_e}{\kappa_B T} > 1 \Rightarrow y < 0.1 + z < 1.9\,, \qquad (13)$$

the last inequality coming from the not too big magnetic field constraint, $z < 1.8$. However, it can be shown that Equation (13) is more stringent than the previous conditions, so that we discard it. Our final set of conditions for the relevance of quantum mechanics is given by Equations (9), (10) and (11). Alternatively, Equation (12) can be chosen since for large densities it is less stringent than (11). However, of course the dimensional analysis we have used is not to be so strictly taken. Also, the transition from the classical scenario to the quantum scenario deserves a more profound treatment.

The conditions for relevant quantum effects in MHD plasmas can be satisfied for the next generation inertial confinement fusion plasmas or perhaps in astrophysical systems like in Jupiter's core. For instance, a suitable set of parameters is given by $n_0 \sim 10^{31} m^{-3}, T \sim 10^6 K, B_0 \sim 3T$. However, our conditions exclude some quantum plasma systems like the atmosphere of white dwarfs, where the density is so high that the plasma becomes collisionless in view of Pauli blocking. Another case where quantum mechanics would be necessary for charged particle system is for very small dimensions, so that $L < \lambda_q$, where L is a typical size of the system. This happens for ultra-small semiconductor devices. Also, the electron gas in metals may be regarded as a quantum plasma system, with the ions in the lattice providing charge neutrality. However, both systems are not described by QMHD, since in QMHD the ions are supposed to be free to move, while in most cases in semiconductor devices or for the electron gas in metals, the ions are attached to a crystalline lattice. In addition, the range of temperatures and densities we have deduced are not attained in typical ultra-small electronic devices. In these systems quantum effects like resonant tunneling will occur more probably in view of the microscopic dimensions.

Some signs of quantum perturbations to MHD were identified in reference 10. For instance, in this work magnetostatic equilibria were found, differing from classical magnetostatic equilibria. As a general rule, QMHD equilibria does not satisfy the classical magnetic surface condition $P + B^2/(2\mu_0) =$ cte., where P is the hydrostatic pressure. Moreover, quantum magnetostatics include a periodic oscillation pattern of density, with no classical counterpart [10]. Similar periodic patterns were also observed in quantum beam instabilities [2]. Therefore, a careful analysis of real MHD magnetostatic equilibria under extreme conditions may be a valuable approach for the identification of departures from classical MHD.

CONCLUSION

Using dimensional analysis, we have derived a set of conditions to be fulfilled for the effectiveness of the QMHD model. A delicate point in the derivation is the assumption $g_q > 1$, assuring a large rate of collisions. Indeed, the quantum graininess parameter g_q is very different from the classical graininess parameter $g_c = e^2 n_0^{1/3}/(\varepsilon_0 \kappa_B T)$. Hence, the condition $g_q > 1$ gives very different results than $g_c > 1$. In particular, quantum plasmas become more collisionless for large densities, in opposition to classical plasmas. In order to justify that $g_q > 1$ indeed imply a large number of collisions, we have argued in terms of the Pauli exclusion principle, implying that two fermions cannot be in the same quantum state. However, the QMHD model does not take into account the statistics of the charge carriers, as seen in the derivation in reference [10]. In fact, in this work the Wigner equation in the presence of magnetic fields was written without regard to any spin properties of the charge carriers. Therefore, the derivation of the collisional constraint (9) should be taken as an additional element, not contained in the framework of the QMHD model. A more complete theory have to start from the Wigner equation for charged fermions in the presence of magnetic fields and possibly including a collision term. However, the Wigner equation for fermions is written in terms of a Wigner matrix whose entries are related to the components of the spinor describing the state of the fermions [12]. In addition, there is no general agreement about the form of the quantum collision operator to be included in the kinetic equation. One possibility in this regard is given by Wigner-Fokker-Planck models [13], which are not, however, adapted to fermionic systems. Therefore, it seems that some effort is to be made to take into account spin and collisions, justifying phenomenological models like QMHD. In this respect, future work on QMHD probably will consider the study of the propagation of linear and nonlinear waves and more general magnetostatic equilibria. In addition, we hope the results of the present communication may be valuable for the identification of real MHD plasmas deserving quantum corrections.

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